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Almost complex structures that model nonlinear geometries

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Abstract. *This article studies a class of connections defined on a symplectic manifold with a Lagrangian foliation that model certain aspects of local differential geometry. This construction is of interest because it provides a more satisfactory treatment of the Equivalence Principle in General Relativity, and it offers a new approach to the study of geometric structures. Homogeneous direction-dependent metrics are studied using these techniques. Conditions are given that guarantee the existence of horizontal distributions that generalize the Levi-Civita and Cartan connections.*

INTRODUCTION

This article studies a class of almost complex connections that model certain local computations of classical differential geometry. Observe the simple fact that a coordinate chart determines a local integrable Lagrangian distribution on the cotangent bundle that is transverse to the vertical. This article introduces a technique that allows the manipulation of such distributions in much the same way that coordinates are used in classical geometry. A fact that is brought out in the following arguments is that the formulas of classical geometry apply to a class of Lagrangian distributions that has as a proper subset the distributions determined by coordinate charts. Thus, from the perspective of this model, it makes sense to interpret Lagrangian distributions as extended coordinate systems.

One reason for studying these structures is that the notion of extended coordi-

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nates clarifies certain foundational problems in General Relativity relating to the equivalence principle. Using this method it is possible to define pseudo-gravitational forces in spaces with non-vanishing curvature. This is not possible within the framework of classical geometry since the definition of a pseudo-gravitational force requires the existence of a complete set of inertial frames. Since inertial motion is defined by geodesic motion, such a set of frames exists from the classical point of view only if the space-time is flat.

Also there are computational advantages in applying these techniques to the study of Finsler connections. It shall be shown that this method allows the identification of a large class of connections that possess the tensorial properties of the Rund and Cartan connections. Also, by this method, it is possible to construct the Rund and Cartan connections for virtually all homogeneous metrics on the vertical distribution of the cotangent bundle.

A number of authors have used almost complex structures as a tool in the study of connections; see [1]. One difference between this and earlier work is that this construction does not use the bundle structure of the cotangent bundle. Geometric structures on the cotangent bundle are only examples of this construction. Many of the results given here apply equally well to any symplectic manifold with a Lagrangian foliation.

1. SYMPLECTIC AND ALMOST COMPLEX CONNECTIONS

Let M be a C^∞-manifold. Denote the ring of C^∞-functions on M by $\mathscr{F}(M)$, the $\mathscr{F}(M)$-module of C^∞-vector fields on M by $\mathscr{X}(M)$, and, in general, the $\mathscr{F}(M)$-module on $C^\infty-(m,n)$-tensors on M by $\mathscr{T}^{(m,n)}(M)$. If X is a distribution on M, then for any $p \in M$, let X_p be the subspace of TM_p determined by X. Let $\mathscr{X}(X)$ be the vector fields on M with values in X, and, in general, let $\mathscr{T}^{(m,n)}(X)$ be the $\mathscr{F}(M)$-module of $C^\infty-(m,n)$-tensors over X. If ω is a nondegenerate differential 2-form on M, then a distribution X is *Lagrangian*, if for every $p \in M$, X_p is a maximally isotropic subspace of TM_p relative to ω_p. A pair of distributions (X, Y) that determine a splitting of TM is a *Lagrangian splitting* if X and Y are Lagrangian distributions.

This section studies certain connections associated with geometric structures determined by the triple $((X, Y), g, \omega)$. Here ω is a nondegenerate differential 2-form on M. (X, Y) is a Lagrangian splitting relative to ω, and g is a metric defined on X. The triple $((X, Y), g, \omega)$ shall be referred to as a *nonlinear geometry*. This terminology is chosen because such geometric structures can be associated with nonlinear connections in tangent bundle.

Example 1.1. The classical example of a geometric structure of this type is Finsler geometry. Here $M = T^*N$ for some C^∞-manifold N, and ω is the canonical symplectic form. X is the vertical distribution. The metric g along X is determined by a homogeneous function of degree one, $f \in \mathscr{F}(T^*N)$. For U, $V \in \mathscr{X}(X)$, such that U and V are parallel relative to the natural affine structure on X, define g at $p \in T^*N$ by $g(U, V)_p = UVf_p^2$. Assume g to be nondegenerate. In classical studies of Finsler geometry, Y is determined by the choice of local coordinates on N.

Example 1.2. Let $M = T^*N$ and let (X, Y) and ω be as in Example 1.1. A metric on N determines by affine translation a metric q along X. Let $g = e^f q$ for some $f \in \mathscr{F}(T^*N)$. It shall be seen that geometric structures of this sort arise naturally from certain parallel transports along curves in N.

The geometric structure $((X, Y), g, \omega)$ determines a pair of (1,1)-tensors (P, J) on M. P is the projection given by the splitting (X, Y) and is chosen so that $\operatorname{ran}(P) = Y$. Let $P^\perp = 1 - P$. J is an almost complex structure determined by the metric along X and by the pairing of X and Y induced by ω. For $u \in Y_p$ and σ, $\omega \in X_p$ define $J : Y_p \to X_p$ by $\omega(u, v)_p = g(Ju, v)_p$ and $J : X_p \to Y_p$ by $\omega(u, Jw)_p = g(u, w)_p$. It is not hard to see that $J \in \operatorname{sp}(TM)$ and $P \in \operatorname{csp}(TM)$ with conformal factor 1. Here as usual $\operatorname{sp}(TM)$ denotes those $A \in \mathscr{T}^{(1,1)}(M)$ that satisfy $\omega(AU, V) + \omega(U, AV) = 0$ for any U, $V \in \mathscr{X}(M)$ and $\operatorname{csp}(TM)$ denotes those $A \in \mathscr{T}^{(1,1)}(M)$ that satisfy $\omega(AU, V) + \omega(U, AV) = k\omega(U, V)$ for some $k \in \mathbb{R}$ and any U, $V \in \mathscr{X}(M)$. The real number k is called the conformal factor of A; see [3] page 117.

The fact that $J \in \operatorname{sp}(TM)$ implies that the metric along X can be extended to a metric on M that shall also be denoted by g. Define for any u, $v \in TM_p$ $g(u, v)_p = \omega(u, Jv)_p$. Note that the orthogonal complement of X_p is Y_p and vice versa.

The following constructions use the standard definition of a linear connection on M. A *linear connection* is a $\mathbb{R}$-bilinear map $\nabla : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ that satisfies for $f \in \mathscr{F}(M)$ (i) $\nabla_{fU} V = f\nabla_U V$ and (ii) $\nabla_U f V = (Uf)V + f\nabla_U V$. This definition can be extended to distributions. Let X be a distribution. A *linear connection along* X is a $\mathbb{R}$-bilinear map $\nabla : \mathscr{X}(X) \times \mathscr{X}(X) \to \mathscr{X}(X)$ satisfying (i) and (ii) for $f \in \mathscr{F}(M)$.

The first useful connection associated with $((X, Y), g, \omega)$ depends only on (X, Y) and ω. It is useful because it provides a background with which to compare the metric connections to be introduced shortly.

PROPOSITION 1.2. *Let* $\overline{\nabla} : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ *be the* $\mathbb{R}$*-bilinear map defined as follows.*

(i) *For* $U \in \mathscr{X}(X)$ *and* $V \in \mathscr{X}(Y)$ *let* $\overline{\nabla}_U V = P[U, V]$ *and* $\overline{\nabla}_V U = P^\perp [V, U]$.

(ii) *For U, $V \in \mathscr{X}(X)$ or U, $V \in \mathscr{X}(Y)$ let Z be defined by* $i(Z)\omega = L_U i(V)\omega$. *If U, $V \in \mathscr{X}(X)$ let* $\bar{\nabla}_U V = P^{\perp}Z$, *or if U, $V \in \mathscr{X}(Y)$ let* $\bar{\nabla}_U V = PZ$.

$\bar{\nabla}$ *is a linear connection in M that satisfies* $\bar{\nabla}\omega = \bar{\nabla}P = 0$.

Proof. See [4]. ■

The connection $\bar{\nabla}$ is commonly referred to as the *Bott connection*. The Bott connection is used to construct generalized affine extensions of tensor fields along the distributions X and Y. A tensor field K is $\bar{\nabla}$*-parallel along* X (or Y) if $\bar{\nabla}_U K = 0$ for all $U \in \mathscr{X}(X)$ (or $\mathscr{X}(Y)$).

The integrability of X and Y and the closure of ω guarantee that the torsion $\bar{T}$ of $\bar{\nabla}$ vanishes and that the $X_p \times X_p$ and $Y_p \times Y_p$ components of the curvature $\bar{R}$ of $\bar{\nabla}$ vanish. However, the $X_p \times Y_p$ components of $\bar{R}$ measure a more subtle relation between (X, Y) and ω. A simple geometric criterion that implies the vanishing of the $X_p \times Y_p$-components of $\bar{R}$ can be stated if it is assumed that $\mathrm{d}\omega = 0$ and that X is integrable. If $f \in \mathscr{F}(M)$, let X_f be the vector field that satisfies $i(X_f)\omega = \mathrm{d}f$. Also denote by $\mathscr{F}(X)$ those $f \in \mathscr{F}(M)$ with the property that $Vf = 0$ for all $V \in \mathscr{X}(X)$.

DEFINITION 1.2. Let X be an integrable Lagrangian distribution of a symplectic manifold and let k be an integer. A *Lagrangian splitting* (X, Y) *is of order* k if for any $U \in \mathscr{X}(Y)$ that is $\bar{\nabla}$-parallel along X, and for any $f^0, \ldots, f^k \in \mathscr{F}(X)$, then $[X_{f^0}, [X_{f^1}, \ldots, [X_{f^k}, U] \ldots]] = 0$.

Example 1.3. Let $M = T^*N$ and let X and ω be as in Example 1.1. If Y is the distribution of a linear connection, then (X, Y) is of order 1 relative to X.

PROPOSITION 1.3. *If X is integrable and* $\mathrm{d}\omega = 0$, *then for $U \in \mathscr{X}(X)$ and* $V \in$ $\in \mathscr{X}(Y)$ $\bar{R}(U, V) = 0$ *if and only if (X, Y) is of order 1.*

Proof. Let f, $g \in \mathscr{F}(X)$ and let $V \in \mathscr{X}(Y)$ such that V is $\bar{\nabla}$-parallel along X. $\bar{R}(X_f, V)X_g = \bar{\nabla}_{X_f}(\bar{\nabla}X_g)(V) - \bar{\nabla}_V(\bar{\nabla}X_g)(X_f) = \bar{\nabla}_{X_f}\bar{\nabla}_V X_g$. For $U \in \mathscr{X}(Y)$, $\omega(\bar{\nabla}_{X_f}\bar{\nabla}_V X_g, U) = L_{X_f}\omega([V, X_g], U) - \omega([X_f, [X_g, V]], U)$. Since $L_{X_f}\omega = 0$, it follows (X, Y) is of order 1, if and only if $\bar{R}(X_f, V) = 0$. ■

When $\bar{\nabla}$ is flat, that is, when $\bar{R} = \bar{T} = 0$, the Lagrangian splitting is said to be *Heisenberg*.

There is a large class of connections associated with the metric g that are of importance here. These connections are constructed from the connections along X and Y given in the next proposition.

PROPOSITION 1.4. *For any* $K \in \mathscr{T}^{(1,2)}(X)$ *(or* $\mathscr{T}^{(1,2)}(Y)$*) there is a connection* D *along* X *(or* Y*) such that for any* $U, V, W \in \mathscr{X}(X)$ *(or* $\mathscr{X}(Y)$*)*

$$(D_U g)(V, W) = \frac{1}{2} g(K(V, W) + K(W, V), U) \tag{1.1}$$

$$T(U, V) = \frac{1}{2}(K(U, V) - K(V, U)), \tag{1.2}$$

where $T(U, V) = D_U V - D_V U - P^{\perp}[U, V]$ *(or* $= D_U V - D_V U - P[U, V]$*).*

Proof. By Christoffel elimination. ∎

When a pair of $(1, 2)$-tensors, $K' \in \mathscr{T}^{(1,2)}(X)$ and $K \in \mathscr{T}^{(1,2)}(Y)$, are given, Proposition 1.4 determines a pair of connections, D' along X and D along Y. This pair can be extended to an almost complex connection on M.

PROPOSITION 1.5. *Given a pair of connections* (D', D) *determined by* (K', K), $K' \in \mathscr{T}^{(1,2)}(X)$ *and* $K \in \mathscr{T}^{(1,2)}(Y)$, *then the* $\mathbb{R}$*-bilinear map* $\nabla : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ *defined by*

(i) *for* $U \in \mathscr{X}(X)$ *and* $V \in \mathscr{X}(Y)$, $\nabla_U V = -JD'_U JV$ *and* $\nabla_V U = -JD_U JU$,

(ii) *for* $U, V \in \mathscr{X}(X)$, $\nabla_U V = D'_U V$ *or for* $U, V \in \mathscr{X}(Y)$, $\nabla_U V = D_U V$,

is a connection on M *that satisfies* $\nabla P = \nabla J = 0$. *Furthermore, if* $K' = K = 0$, *then* $\nabla \omega = 0$.

Proof. Similar to Proposition 1.2. ∎

The connection ∇ shall be called the *almost complex connection for* $((X, Y), g, \omega)$ *defined by* (K', K).

A very useful relation exists between the torsion T of ∇ and the difference tensor $\bar{S} = \bar{\nabla} - \nabla$. Clearly, for $U \in \mathscr{X}(X)$ and $V \in \mathscr{X}(Y)$, $\bar{S}(U, V) = PT(U, V)$ and $\bar{S}(V, U) = P^{\perp}T(V, U)$. However, PT and $P^{\perp}T$ also determine the components of $\bar{S}$ in the $X_p \times X_p$ and $Y_p \times Y_p$ directions.

PROPOSITION 1.6. *Let* ∇ *be the almost complex connection defined by the pair* (K', K), *and let* $\bar{S} = \bar{\nabla} - \nabla$. *For* $U, V, W \in \mathscr{X}(X)$

$$\omega(W, PT(V, JU)) = \omega(U, J\bar{S}(V, W)) - \frac{1}{2} g(K'(U, W) + K'(W, U), V) \tag{1.3}$$

and for $U, V, W \in \mathscr{X}(Y)$

$$\omega(W, P^{\perp}T(V, JU)) = \omega(U, J\overline{S}(V, W) - \tag{1.4}$$
$$- \frac{1}{2}\, g(K(U, W) + K(W, U), V).$$

Proof. The proof is a computation that follows from the definition of ∇ and $\overline{\nabla}$. ∎

Example 1.4. Return to Example 1.2. Since the Bott connection along X is the torsion-free metric connection along X induced by q, it follows that the torsion-free metric connection along X induced by g is conformally related to the Bott connection. This fact and (1.4) imply that for $V \in \mathscr{X}(X)$ and $U \in \in \mathscr{X}(Y)$

$$PT(V, U) = \frac{1}{2}\,(-(JUf)\,JV - (Vf)\,U + \omega(U, V)\,J\,\nabla f),$$

where ∇f is the gradient of f in the metric g.

An obvious and important choice of the (1, 2)-tensors (K', K) is to set $K' = = K = 0$. In this case Proposition 1.5 implies that $\nabla g = 0$. This restriction also has the implication that the torsion T of ∇ determines the curvature of R of ∇. To see this, recall the first Bianchi identity with torsion. Let $U, V, W \in \mathscr{X}(M)$ and let $\mathscr{S}$ denote the sum over the cyclic permutations of U, V, W. The first Bianchi identity states that

$$\mathscr{S}(R(U, V)W) = \mathscr{S}((\nabla_U T)(V, W) - T(U, T(V, W))). \tag{1.5}$$

For notational conveneince define $Q \in \mathscr{T}^{(0,4)}(M)$ by $Q(U, V, W, Z) = \omega(\mathscr{S}((\nabla_U T)\cdot \cdot (V, W) - T(U, T(V, W))), Z)$. The next proposition shows that all the components of R can be expressed in terms of Q, P and J.

PROPOSITION 1.7. *Let* ∇ *be the almost complex connection defined by* (0, 0).

(i) *For* $U, V \in \mathscr{X}(X)$ *and any* $Z, W \in \mathscr{X}(M)$

$$\omega(R(U, V)Z, W) = Q(U, V, PZ, P^{\perp}W) + Q(U, V, PW, P^{\perp}Z), \tag{1.6}$$

and for $U, V \in \mathscr{X}(Y)$ *and any* $Z, W \in \mathscr{X}(M)$

$$\omega(R(U, V)Z, W) = Q(U, V, P^{\perp}Z, PW) + Q(U, V, P^{\perp}W, PZ). \tag{1.7}$$

(ii) *For* $U \in \mathscr{X}(X)$ (*or* $\mathscr{X}(Y)$) *and* $V, Z, W \in \mathscr{X}(Y)$ (*or* $\mathscr{X}(X)$)

$$\begin{aligned}\omega(R(U,V)W,JZ) &= \frac{1}{2}(Q(U,Z,V,JW) + \\ &+ Q(U,V,W,JZ) + Q(U,Z,W,JV)).\end{aligned} \tag{1.8}$$

Proof. The proof is a computation based upon the isotropy of X and Y, (1.5), and the symmetry properties of R, that is, $\omega(R(U,V)W,Z) = \omega(R(U,V)Z,W)$ and $g(R(U,V)W,Z) = -g(R(U,V)Z,W)$. ∎

2. DIFFERENCE TENSORS

This section studies the relationship between connections determined by $((X,Y),g,\omega)$ and connections determined by $((X,Y'),g,\omega)$. For the rest of this article it shall be assumed that X *is integrable*, and if ∇ *is an almost complex connection for* $((X,Y),g,\omega)$ *defined by* (K',K), *then* $K' = 0$. These assumptions are not essential, but to proceed without them would be cumbersome. In any event, the important examples satisfy these conditions. An immediate consequence of these restrictions is that the torsion of ∇ and $\overline{\nabla}$ vanishes along X. This implies that, for any $p \in M$, $\overline{S} = \overline{\nabla} - \nabla$ is symmetric in the X_pXX_p--direction, and so by (1.3) for $U, V, W \in \mathscr{X}(X)$

$$\omega(W,PT(V,JU)) = \omega(V,PT(W,JU)). \tag{2.1}$$

Let Y and Y' be distributions transverse to X. There exists a unique $A \in \mathscr{T}^{(1,1)}(M)$ such that $\operatorname{ran}(A) \subseteq X \subseteq \ker(A)$ and for all $u \in Y_p$, $u + A_pu \in Y'_p$. A shall be called the graph coordinate of the ordered pair (Y,Y'). Note that if Y and Y' are Lagrangian distributions $A \in \operatorname{sp}(TM)$. Let (P,J) and (P',J') be the projections and complex structures defined by $((X,Y),g,\omega)$ and $((X,Y'),g,\omega)$. It can be seen that $P' = P + A$ and $P'^{\perp} = P^{\perp} - A$, and that $J' : X \to Y'$ is given by $J' = J + AJ$ and $J' : Y' \to X$ is given by $J' = JP$.

Example 2.1. Let $M = T^*N$ and let $\pi : T^*N \to N$ be the canonical fibration. Suppose that (x,U) and (x',U) are charts on T^*N derived from charts $(y,\pi(U))$ and $(y',\pi(U))$ on N. Let $\pi_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the projection onto the first factor, and suppose that $\pi_0 \circ x = y \circ \pi$. Let (X_0, Y_0) be the canonical splitting of $T\mathbb{R}^n \times \mathbb{R}^n$ given by, for $(t,s) \in \mathbb{R}^n \times \mathbb{R}^n$, $X_{0(t,s)} = 0 \times \mathbb{R}^n$ and $Y_{0(t,s)} = \mathbb{R}^n \times 0$, and let P_0 be the associated projection with $\operatorname{ran}(P_0) = Y_0$. Define the distributions Y and Y' on U by $Y = x^{-1}{}_*Y_0$ and $Y' = x'^{-1}{}_*Y_0$. To compute A for (Y,Y'), compute A_0 for the ordered pair $(Y_0, (x \circ x'^{-1})_*Y_0)$. Recall that for $(t,s) \in x(U)$

$$x \circ x'^{-1}(t,s) = (y \circ y'^{-1}(t), sD(y' \circ y^{-1})(t)),$$

$Df : T\mathbb{R}^n \to T\mathbb{R}^m$ is the Jacobian of $f : \mathbb{R}^n \to \mathbb{R}^m$. For $p \in U$, let $x(p) = (t, s)$ and let $u \in Y_{0(t,s)}$ be $u = (t, s, z, 0)$. Then

$$A_{0(t,s)}u = (t, s, 0, D(sD(y' \circ y^{-1})D(y' \circ y^{-1})z),$$

and so $A_p = x_*^{-1} A_{0x(p)} x_*$.

PROPOSITION 2.1. *Let ∇ be the almost complex connection for $((X, Y), g, \omega)$ defined by $(0, K)$ and let ∇' be the almost complex connection for $((X, Y'), g, \omega)$ defined by $(0, K')$. If (P, J) and (P', J') are the projections and almost complex structures defined by $((X, Y), g, \omega)$ and $((X, Y'), g, \omega)$, and A is the graph coordinate of (Y, Y'), then $S = \nabla' - \nabla$ satisfies*

(i) $S(U, V) = 0$ *for* $U, V \in \mathscr{X}(X)$
(ii) $S(V, U) = -(\nabla_V A)(U)$ *for* $V \in \mathscr{X}(X)$, $U \in \mathscr{X}(Y)$
(iii) $S(U, V) = -JPS(U, JV)$ *for* $V \in \mathscr{X}(X)$, $U \in \mathscr{X}(Y)$
(iv) $P^{\perp}S(U, V) = (\nabla_U A)(V) - JPS(U, JAV) - APS(U, V)$
for $U, V \in \mathscr{X}(Y)$.

Further, if for any $U, V, W \in \mathscr{X}(Y)$, $L(U, V) = PK'(P'U, P'V) - K(U, V)$ *and*

$$\begin{aligned} M(U, V) = \frac{1}{2}(&PT(AV, U) - PT(AU, V) + \\ &+ JAPT(JV, U) + JAPT(JU, V) - \\ &- J\bar{S}(AU, JV) - J\bar{S}(AV, JU)), \end{aligned} \tag{2.2}$$

then

$$\begin{aligned} g(W, PS(U, V)) = \frac{1}{2}(&g(L(U, V), W) - \\ &- g(L(U, W), V) - g(L(V, W), U)) + \\ &+ g(W, M(U, V)). \end{aligned} \tag{2.3}$$

Proof. Denote the torsion of ∇ by T and the torsion on ∇' by T'. (i) is clear. (ii) follows from (i), the definition of ∇, and the formulas for J'. To see (iii) first observe the following. (i) and (1.3) imply that for $V, W \in \mathscr{X}(X)$ $PT'(V, JW) = PT(V, JW)$. Since $S(V, U) - S(U, V) = T'(V, U) - T(V, U)$, it follows that for $V \in \mathscr{X}(X)$ and $U \in \mathscr{X}(Y)$ $PS(V, U) - PS(U, V) = 0$. By (ii) $PS(V, U) = 0$, and so $PS(U, V) = 0$. Now a computation shows that $S(U, V) = -JPS(U + AU, JV + AJV)$. However, (i) implies $PS(AU, AJV) = 0$, (ii) implies $PS(AU, JV) = 0$, and the preceding observation implies $PS(U, AJV) = 0$. Therefore, $S(U, V) = -JPS(U, JV)$.

To calculate $PS(U, V)$ for $U, V \in \mathscr{X}(Y)$, note that $P\nabla' : \mathscr{X}(Y) \times \mathscr{X}(Y) \to \mathscr{X}(Y)$ is a connection along Y. Also observe that since $(\nabla'_V g)(P'U, P'W) = 0$ for $V \in \mathscr{X}(X)$ and $U, W \in \mathscr{X}(Y)$, $P\nabla'$ satisfies for $U, V, W \in \mathscr{X}(Y)$

$$(P\nabla'_U g)(V, W) = \frac{1}{2} g(PK'(P'V, P'W) + PK'(P'W, P'V), U).$$

This observation implies that $P\nabla'$ along Y is defined by $\hat{K} \in \mathscr{T}^{(1,2)}(Y)$,

$$\hat{K}(W, V) = \frac{1}{2}(PK'(P'W, P'V) + PK'(P'V, P'W)) + PT'(W, V).$$

But, $PT'(W, V) = PT'(P'W, P'V) - PT'(P'W, AV) - PT'(AW, P'V)$, and also (1.2) implies $PT'(P'W, P'V) = \frac{1}{2}(PK'(P'W, P'V) - PK'(P'V, P'W))$. Substituting these expressions into the expression for $\hat{K}$ gives

$$\hat{K}(W, V) = PK'(P'W, P'V) - PT'(P'W, AV) - PT'(AW, P'V).$$

It now follows by Christoffel elimination that the difference between ∇, and ∇' along Y is given by (2.4). Here the fact that for $V \in \mathscr{X}(X)$ and $U \in \mathscr{X}(Y)$ $PT'(V, P'U) = PT(V, U)$ is used to eliminate T' from the expression for $\hat{K}$. Also note that (2.1) and (1.4), imply taht for $U, V, W \in \mathscr{X}(Y)$ $g(PT(AU, W), V) = -g(J\overline{S}(AU, JV), W)$ and $g(PT(AW, U), V) = -g(JAPT(JV, U), W)$. ∎

Proposition 2.1 implies the following statement about the dependence of the X_p-component of the torsion of an almost complex connection on the choice of Y. Notice that if X, g, and ω remain fixed the Y_p-component of the torsion is independent of Y.

PROPOSITION 2.2. *Using the notation of Proposition 2.1, let T be the torsion of ∇ and let T' be the torsion of ∇'. For $V \in \mathscr{X}(X)$ and $U \in \mathscr{X}(Y)$,*

$$(2.4) \qquad P'^{\perp} T'(V, U) - PT(V, U) = JPS(U, JV) - (\nabla_V A)(U) - APT(V, U).$$

Proof. This follows from (ii) and (iii) of Proposition 2.1. ∎

(2.4) can be written in the following equivalent form.

PROPOSITION 2.3. *With the notation of Proposition 2.1, let $\overline{\nabla}$ be the Bott connection for $((X, Y), g, \omega)$ and let $\overline{\nabla}'$ be the Bott connection for $((X, Y'), g, \omega)$.*

If $\bar{S} = \bar{\nabla} - \nabla$ *and* $\bar{S}' = \bar{\nabla}' - \nabla'$ *then for any* U, $W \in \mathscr{X}(Y)$ *and* $V \in \mathscr{X}(X)$.

$$\begin{aligned} \omega(V, \bar{S}'(P'U, P'W) - \bar{S}(U, W)) = &- \omega(W, (\nabla_V A)(U)) - \\ &- \omega(V, N(U, W)) + \frac{1}{2}(g(L(U, JV), W) - \\ &- g(L(U, W), JV) + g(L(W, JV), W)) \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} N(U, W) = \frac{1}{2}(&PT(AU, W) + PT(AW, U) + JAPT(JW, U) + \\ &+ JAPT(JU, W) - J\bar{S}(AU, JW) - J\bar{S}(AW, JU)). \end{aligned}$$

Proof. The proof is a rather long computation that relies upon (2.1), (1.3) and (1.4). ∎

If one applies Proposition 2.3 to the special choice of Y and Y' given in Example 2.1, one notes the similarity between (2.5) and the classical expression of the variation of the Christoffel symbols under a change of coordinates. For instance, see [5]. What this suggests is that computations involving classical transformation formulas can be extended to arbitrary Lagrangian distributions transverse to the vertical. Notice that in deriving (2.5) the cotangent condition $d\omega = 0$ was not required. However, this condition does arise when one computes the difference between the Bott connections determined by $((X, Y), g, \omega)$ and $((X, Y'), g, \omega)$.

PROPOSITION 2.4. *Let* $\bar{\nabla}$ *be the Bott connection for* $((X, Y), g, \omega)$ *and let* $\bar{\nabla}'$ *be the Bott connection for* $((X, Y'), g, \omega)$. *If* $\hat{S} = \bar{\nabla}' - \bar{\nabla}$, *then*

(i) $\hat{S}(U, V) = 0$ *for* U, $V \in \mathscr{X}(X)$
(ii) $\hat{S}(V, U) = -(\bar{\nabla}_V A)(U)$ *for* $V \in \mathscr{X}(X)$, $U \in \mathscr{X}(Y)$
(iii) $\hat{S}(U, V) = -(\bar{\nabla}_V A)(U)$ *for* $U \in \mathscr{X}(X)$, $V \in \mathscr{X}(Y)$
(iv) $\omega(W, P\hat{S}(U, V)) = d\omega(AU, W, V) - \omega(V, \bar{\nabla}_W A(U))$ *for* $W \in \mathscr{X}(X)$, U, $V \in \mathscr{X}(Y)$

Proof. The proof uses the same techniques as Proposition 2.1 but is easier. ∎

3. COVARIANT DERIVATIVES

From the point of view of a classical geometer, covariant geometric objects are those objects whose transformation laws do not involve second derivatives of a coordinate change. To introduce an analogous concept into the present framework, note that each Heisenberg splitting (X, Y) of TT^*N, where X is the vertical distribution, corresponds to an orbit of charts on N under the action of the affine group on $\mathbb{R}^n$. Therefore, in view of Example 2.1, a natural extension of the classical definition of covariance is to say that a collection of geometric objects is covariant if each object is independent of the choice of Y used in its construction.

This section studies a particular covariant object, namely the covariant derivative. Along with the restrictions on $((X, Y), g, \omega)$ introduced in Section 2., in this section assume that $d\omega = 0$.

DEFINITION 3.1. Let ∇ and ∇' be almost complex connections for $((X, Y), g, \omega)$ and $((X, Y'), g, \omega)$ respectively, and let $\bar{\bar{\nabla}}$ be the Bott connection for $((X, Y), g, \omega)$

(i) ∇ and ∇' are *covariantly related* if for any $U, V \in \mathscr{X}(Y)$, such that V is $\bar{\bar{\nabla}}$-parallel along X, then $P\nabla'_{P'U}P'V = \nabla_U V$.

(ii) ∇ and ∇' are *semi-covariantly related* if for any $U, V \in \mathscr{X}(Y)$, then $P\nabla'_{P'U}P'V = \nabla_{P'U}V$.

Definition 3.1(i) and 3.1(ii) determine the two most important classes of connections associated with nonlinear geometries. In Finsler geometry the Rund connection is equivalent to a set of covariantly related almost complex connections; while the Cartan connection is equivalent to a set of semi-covariantly related almost complex connections. The following proposition constructs a generalization of these objects that is adapted to the present setting.

DEFINITION 3.2. Let (X, Y'') be a fixed Lagrangian splitting of TM. For a Lagrangian splitting (X, Y), define $K_Y \in \mathscr{T}^{(2,1)}(Y)$ and $\hat{K}_Y \in \mathscr{T}^{(2,1)}(Y)$ by

$$K_Y(U, V) = JAPT(JU, V) + JAPT(JV, U), \tag{3.1}$$

$$\hat{K}_Y(U, V) = PT(AV, U) - PT(AU, V). \tag{3.2}$$

Here A is the graph coordinate of (Y, Y''), (P, J) is determined by $((X, Y), g, \omega)$, and T is the torsion of the almost complex connection for $((X, Y), g, \omega)$ defined by $(0, 0)$.

PROPOSITION 3.1. *If ∇ is the almost complex connection for $((X, Y), g, \omega)$ defined by $(0, K_Y)$ (or $(0, \hat{K}_Y)$) and ∇' is the almost complex connection for $((X, Y'), g, \omega)$ defined by $(0, K_{Y'})$ (or $(0, \hat{K}_{Y'})$), then ∇ and ∇' are (semi-)*

covariantly related.

Proof. Let C be the graph coordinate of (Y', Y''), let B be the graph coordinate of (Y, Y''), and let A be the graph coordinate of (Y, Y'). The proof is based on (2.5). The first step is to compute $L(U, V) = PK_{Y'}(P'U, P'V) - K_Y(U, V)$ for $U, V \in \mathscr{X}(Y)$. The facts that $PJ'CP'T'(J'P'U, P'V) = JCP'PT(JU, V)$, and that $CP' = B - A$ imply

$$PK_{Y'}(P'U, P'V) - K_Y(U, V) = -JAPT(JU, V) + JAPT(JV, U).$$

Using (1.3), a calculation shows that in (2.5) the terms in PT and $\bar{S}$ from L cancel the terms in PT and $\bar{S}$ from N, and so (2.5) reduces to

$$\omega(V, \bar{S}'(P'U, P'W) - \bar{S}(U, W)) = -\omega(W, (\bar{\nabla}_V A)(U)). \tag{3.3}$$

Since $d\omega = 0$, Proposition 2.4(iv) implies that $\omega(V, P\hat{S}(U, W)) = -\omega(W, (\bar{\nabla}_V A) \cdot \cdot (U))$. Now if W is $\bar{\nabla}$-parallel along X, it is not hard to see that $P'W$ is $\bar{\nabla}'$-parallel along X. Therefore,

$$P\hat{S}(U, W) = P(\bar{\nabla}'_U W - \bar{\nabla}_U W) = P(\bar{\nabla}'_{U+AU} W + AW - \bar{\nabla}_U W).$$

Substituting this expression into (3.3) gives the desired result. The proof that ∇ and ∇' are semi-covariantly related when ∇ and ∇' are determined by $(0, \hat{K}_Y)$ and $(0, \hat{K}_{Y'})$ is similar. ∎

An important fact used in the proof of Proposition 3.1 is that the graph coordinates A, B, and C of the pairs (Y, Y'), (Y, Y''), and (Y', Y'') satisfy $CP' = B - A$. Geometrically this formula is a generalization of the first derivative of the cocycle condition on coordinate charts. Physically it is analogous to the addition law of velocities in Galilean Relativity. Also, note that the method used in Proposition 3.1 to construct covariantly related connections is in spirit similar to the techniques used in classical geometry to construct covariant derivatives. It can be shown that the above connections are the only linear connections constructed from PT and A that satisfy Definition 3.1.

4. HOMOGENEITY CONDITIONS

This section studies the effects of homogeneity criteria on nonlinear geometries. In this section as in Section 3 assume $d\omega = 0$. To introduce the notion of homogeneity into this construction, fix once and for all an integrable Lagrangian distribution X and a 1-form α that satisfies (i) $\alpha(V) = 0$ for $V \in \mathscr{X}(X)$ and (ii) $d\alpha = \omega$. The existence of X and α implies that if the leaves of X are absolutely parallelizable, then M is symplectically equivalent to a cotangent bundle. Let

X_α be the vector field along X defined by $i(X_\alpha)\omega = \alpha$. The fact that X_α acts as a homogeneity operator is suggested by the fact that $L_{X_\alpha}\omega = \omega$.

DEFINITION 4.1. Let k be an integer. A pair of Lagrangian distributions (Y, Y'), each tranverse to X, is said to be *homogeneous of degree* k if the graph coordinate A of (Y, Y') satisfies $\overline{\nabla}_{X_\alpha} A = kA$, where $\overline{\nabla}$ is the Bott connection defined by (X, Y) and ω.

PROPOSITION 4.1. *Homogeneity of degree k is an equivalence relation on the set of Lagrangian distributions transverse to X.*

Proof. This is a consequence of Proposition 2.4. ∎

If the homogeneity relation is of degree 1, it is possible to label the equivalence classes in the following sense.

PROPOSITION 4.2. *Let (Y, Y') is homogeneous of degree 1. If $\overline{\nabla}$ and $\overline{\nabla}'$ are the Bott connections defined by Y and Y' and if P' is the projection defined by Y', then $\overline{\nabla}_U X_\alpha = \overline{\nabla}'_{P'U} X_\alpha$ for $U \in \mathscr{X}(Y)$.*

Proof. It is not hard to see that X_α has the property that $\overline{\nabla}_V X_\alpha = V$ for $V \in$ $\in \mathscr{X}(X)$. The conclusion follows from this fact and Proposition 2.4(iii). ∎

Proposition 4.2 says that the 1-form $\overline{\nabla} X_\alpha$ is pointwise constant when evaluated along equivalent degree 1 Lagrangian distributions. In particular, if $M = T^*N$ and α is the canonical 1-form, then $\overline{\nabla} X_\alpha = 0$ along the elements of the degree 1 equivalence class that contains the coordinates. In general, call the degree 1 equivalence class with this property the *coordinate class. Denote the coordinate class by* $\mathscr{Y}_0$.

Example 4.1. In the case where $M = T^*N$, $\mathscr{Y}_0$ contains elements that are not defined by coordinate charts on N. Let α be the canonical 1-form and let ϕ_t be the flow of X_α. Let Y be a Lagrangian integrable distribution. If Y is homogeneous, that is $\phi_{t*} Y = Y$, then $Y \in \mathscr{Y}_0$. A local integral manifold of Y can be locally identified with a closed 1-form on N. 1-forms determined by Y are of interest in relativity because they correspond to the local time functions of synchronous observers. A set of synchronous observers shall be called *complete* if the corresponding set local time functions locally foliate the light cone. In a flat space-time, constant time-like linear combinations of the coordinate functions determine a complete set of synchronous observers.

DEFINITION 4.2. $((X, Y), g, \omega)$ is said to be *semi-projectable* if there exists an $r \in \mathbb{R}$ such that for $U \in \mathscr{X}(Y)$

$$PT(X_\alpha, U) = rU, \tag{4.1}$$

where T is the torsion of the almost complex connection defined by $(0, 0)$.

Definition 4.2 is a computationally efficient homogeneity condition on $((X, Y), g, \omega)$. It is equivalent to the following more natural criterion.

PROPOSITION 4.3. *$((X, Y), g, \omega)$ is semi-projectable if and only if the almost complex connection for $((X, Y), g, \omega)$ defined by $(0, 0)$ satisfies for $V \in \mathscr{X}(X)$ $\nabla_V X_\alpha = (1 - r)V$.*

Proof. Easy consequence of (4.1) and (1.3). ∎

Proposition 4.3 shows that semi-projectability is a property of the metric, and is independent of the choice of Y. The notion of a semi-projectable triple is weaker than similar conditions given in [2] and [6] where it is assumed that $r = 0$.

Example 4.2. If PT is as in Example 1.4. and if PT is semi-projectable, then there is $k \in \mathbb{R}$ so that $f = \log(\rho^k)$ where $\rho = |q(X_\alpha, X_\alpha)|^{1/2}$. If $q(X_\alpha, X_\alpha) < 0$, then

$$PT(V, U) = \frac{k}{2\rho^2}\,(q(JU, X_\alpha)JV + q(V, X_\alpha)U - q(JU, V)Z), \tag{4.2}$$

and so $r = -(k/2)$.

Given a triple $((X, Y), g, \omega)$ and the homogeneity operator X_α, the vector field $Z = JX_\alpha$ is called the *dynamical vector field determined by* $((X, Y), g, \omega)$.

Example 4.3. Let $M = T^*N$ and let α be the canonical 1-form. If X is the vertical distribution, for $p \in T^*N$ let $i_p : X_p \to T^*N_{\pi(p)}$ be the natural identification, and let $j_p : TN_{\pi(p)} \to X_p^*$ be dual to i_p. If g is a metric along X, then g induces a Legendre transformation $l : T^*N \to TN$ as follows. For $p \in T^*N$ define $\lambda_p \in X_p^*$ by $\lambda_p \doteq i(X_\alpha)g_p$. Let $l(p) = j_p^{-1}\lambda_p$. For any $Z \in \mathscr{X}(T^*N)$ that satisfies $g(X_\alpha, U) = -\omega(Z, U)$ for all $U \in \mathscr{X}(X)$, it can be seen that $\pi_* Z = l$. Further, if Y is integrable, then a vector field, that corresponds to the image of a leaf of Y under the Legendre transformation l, is a geodesic vector field for the second order system determined by Z. If $Y \in \mathscr{Y}_0$, this set of vector fields can be viewed as the coordinate directions of the observers whose time functions are determined by Y.

When a triple $((X, Y), g, \omega)$ is semi-projectable there is a special relation between Z and the associated connection theory. Suppose that $Y \in \mathcal{Y}_0$, then $\nabla_Z Z = J\nabla_Z X_\alpha = JP^\perp T(X_\alpha, Z)$, It shall be shown that if $r \neq 1$, then there exists $Y'' \in \mathcal{Y}_0$ such that if C is the graph coordinate of (Y, Y''), then

$$\nabla_Z Z = (1-r)JCZ. \tag{4.3}$$

Note that this expression has the form of a force law scaled by the real number $(1-r)$.

LEMMA 4.1. *If* $Y \in \mathcal{Y}_0$ *and if* $((X, Y), g, \omega)$ *is semi-projectable, then for any* $V \in \mathcal{X}(X)$ *and* $U \in \mathcal{X}(Y)$ $(\overline{\nabla}_{X_\alpha} P^\perp T)(V, U) = 0$.

Proof. Let $U \in \mathcal{X}(Y)$ be $\overline{\nabla}$-parallel along X and let $V \in \mathcal{X}(X)$ be $\overline{\nabla}$-parallel along Y. First show that $\overline{\nabla}_U X_\alpha = 0$ implies that $\overline{R}(X_\alpha, U)V = 0$. Since $\overline{R}(X_\alpha, U)V = P^\perp[U, [X_\alpha, W]]$ and $[U, V] \in \mathcal{X}(Y)$ and $[X_\alpha, U] \in \mathcal{X}(Y)$, this follows by the Jacobi identity. Next express $\overline{R}$ in terms of R and the difference tensor $\overline{S} = \overline{\nabla} - \nabla$. Using the fact that (4.1) implies that $S(X_\alpha, V) = rV$, a computation shows

$$\overline{R}(U, X_\alpha)V = R(U, X_\alpha)V + (\nabla_{X_\alpha} P^\perp T)(U, V) + rP^\perp T(U, V).$$

Also, $Q(X_\alpha, V, U, JW) = 0$ and so by (1.9) $R(U, X_\alpha)V = 0$. Therefore $(\nabla_{X_\alpha} P^\perp T) \cdot (U, V) = -rP^\perp T(U, V)$, and a change of connection implies that $(\overline{\nabla}_{X_\alpha} P^\perp T) \cdot (U, V) = 0$. ∎

PROPOSITION 4.4. *Let* $((X, Y), g, \omega)$ *be semi-projectable with* $r \neq 1$ *and* $Y \in \mathcal{Y}_0$. *There exists* $Y' \in \mathcal{Y}_0$ *independent of the choice of* Y *such that if* C *is the graph coordinate of* (Y, Y'), *and if* T *is the torsion of the almost complex connection for* $((X, Y), g, \omega)$ *defined by* $(0, 0)$, *then* $P^\perp T(X_\alpha, Z) = (1-r)CZ$.

Proof. Let $Y \in \mathcal{Y}_0$ and let H be a hypersurface transverse to the flow of X_α. For $p \in H$ let $B_p \in \mathrm{sp}(TM)_p$, with $\mathrm{ran}(B_p) \subseteq X \subseteq \ker(B_p)$, such that $BZ_p = P^\perp T(X_\alpha, Z)_p$. Extend B to M by $\overline{\nabla}$-translation along the flow of X_α such that $\overline{\nabla}_{X_\alpha} B = B$. It follows from Lemma 4.1 that $BZ - P^\perp T(X_\alpha, Z)$ is a solution to the o.d.e. $\overline{\nabla}_{X_\alpha} U = 2(1-r)U$, and so $BZ = P^\perp T(X_\alpha, Z)$ everywhere. Define Y' so that $C = (1/(1-r))B$ is the graph coordinate of (Y, Y'). For any other $Y'' \in \mathcal{Y}_0$ let S be the difference between the almost complex connections ∇ and ∇'' for $((X, Y), g, \omega)$ and $((X, Y''), g, \omega)$ defined by $(0, 0)$, and let T'' be the torsion of ∇''. Then (4.1) implies that $S(Z, Z) = 0$ and so (2.4) gives $(1/(1-r)) \cdot \cdot P''^{\perp} T''(X_\alpha, Z') = CZ - AZ = C''Z''$ where A and C'' are the graph coordinates of (Y, Y'') and (Y'', Y'). ∎

DEFINITION 4.3. A semi-projectable triple is *geodesible* if Z is a geodesic vector field.

Note that (2.4) implies that if Y' is the distribution constructed in Proposition 4.4, then $((X, Y'), g, \omega)$ is geodesible. If $r = 1$, (2.4) implies that $P^{\perp}T(X_\alpha, Z)$ is independent of the choice of Y. Consequently, in this case, if there is an element of the coordinate class for which Z is geodesic, then for every element of the coordinate class Z is geodesic. In the case of geometries of Example 4.3 with $r = 1$, it can be seen from (4.2) that for a curve γ on N, parallel translation along $\overline{\gamma} = l^{-1}(\dot{\gamma})$ projects to Fermi translation along γ.

Example 4.4. In relativity a geodesible triple $((X, Y'), g, \omega)$ with $Y' \in \mathscr{Y}_0$ and Y' integrable represents a complete set of inertial observers. If Z is the dynamical vector field determined by a semi-projectable triple $((X, Y), g, \omega)$ with $Y \in \mathscr{Y}_0$, then (4.3) expresses the pseudo-gravitational force observed by Y, and therefore gives a representation of the equivalence principle. When $r = 0$ and g is the flat metric, Example 2.1 shows that this relation reduces to the usual representation of the equivalence principle in flat space. When $r = 0$ and g is not flat, (4.3) is still valid. However, a complete set of inertial observes can no longer be associated with a single chart. In this case Y' is a distribution whose integral submanifolds are solutions to the time independent Hamilton-Jacobi equations determined by g.

If $((X, Y), g, \omega)$ is semi-projectable and $Y \in \mathscr{Y}_0$, then under certain conditions it is possible to construct from the torsion of the almost complex connection defined by $(0, 0)$ the graph coordinate of a Lagrangian distribution $Y' \in \mathscr{Y}_0$ such that $((X, Y'), g, \omega)$ is geodesible. To see this, introduce the following notation, For $L \in \mathscr{T}^{(2,2)}(M)$ define $L : \mathscr{T}^{(1,1)}(M) \to \mathscr{T}^{(1,1)}(M)$ by $L(C) = \mathscr{C}(L \otimes C)$ where $\mathscr{C}$ is the contraction of C on the second and third entries of L.

Note that $I^2 = I \otimes I$ is the identity, and if $A \in \mathscr{T}^{(1,1)}(M)$, then $I \otimes A(C) = CA$ and $A \otimes I(C) = AC$.

PROPOSITION 4.6. *For $Y \in \mathscr{Y}_0$, suppose that $((X, Y), g, \omega)$ is semi-projectable with $r \neq 1$ and let T be the torsion of the almost complex connection defined by $(0, 0)$. Define $E \in \mathscr{T}^{(1,1)}(M)$ by $EU = (1/2(1-r))PT(U, Z)$ and $\Gamma \in \mathscr{T}^{(1,1)}(M)$ by*

$$\Gamma U = \frac{1}{2(1-r)} P^{\perp}T(X_\alpha, U) +$$

$$+ \frac{1}{2(1-r)^2} [JPT(P^{\perp}T(X_\alpha, Z), U) + \overline{S}(P^{\perp}T(X_\alpha, Z), JU)],$$

and let $H = I \otimes EJ + JE \otimes I$. (i) *If* $(3r-2)/(2r-2)$ *is not an eigenvalue of* JE *and* (ii) *if the difference between any two eigenvalues of* JE *is never equal to 1, then* $I^2 + H$ *is invertible and there exists a Lagrangian distribution* $Y' \in \mathscr{Y}_0$ *such that* $((X, Y'), g, \omega)$ *is geodesible and* $C = (I^2 + H)^{-1}(\Gamma)$ *is the graph coordinate of* (Y, Y').

Proof. (ii) implies $I^2 + H$ is invertible and so C is well defined. To verify that if Y' is defined relative to Y by C, then $((X, Y'), g, \omega)$ is geodesible and $Y' \in \mathscr{Y}_0$, one must show that (1) $C \in \mathrm{sp}\,(TM)$, (2) $\overline{\nabla}_{X_\alpha} C = C$, and (3) $(1/(1-r))\, P^{\perp}T(X_\alpha, Z) = CZ$. (1) follows since (1.3) and (1.4) imply that $\Gamma \in \mathrm{sp}\,(TM)$ and that $\mathrm{sp}\,(TM)$ is an invariant subspace for $I^2 + H$. To prove (2) note that Proposition 4.3 implies that for $U, V \in \mathscr{X}(X)$ $R(U, V)X_\alpha = 0$. Applying block symmetry along X, this is equivalent to $R(X_\alpha, U) = 0$ for $U \in \mathscr{X}(X)$. But, (1.6) and $R(X_\alpha, U) = 0$ imply that for $U \in \mathscr{X}(Y)$ and $V \in \mathscr{X}(X)$ $(\overline{\nabla}_{X_\alpha} PT)(V, U) = -PT(V, U)$. This result implies that $\overline{\nabla}_{X_\alpha} \Gamma = \Gamma$ and $\overline{\nabla}_{X_\alpha}(I^2 + H) = 0$, and so $\overline{\nabla}_{X_\alpha} C = C$. (3) follows from (i), as (i) implies that $CZ = (1/(1-r))P^{\perp}T(X_\alpha, Z)$ is the unique solution to $CZ + H(C)Z = \Gamma Z$. Finally, if $Y'' \in \mathscr{Y}_0$ and if $I^2 + H''$ and Γ'' are constructed from the torsion of the almost complex connection for $((X, Y''), g, \omega)$ defined by $(0, 0)$, then (2.4) implies that if C'' is the graph coordinate of (Y'', Y'), then $(I^2 + H'')(C'') - \Gamma'' = (I^2 + H)(C) - \Gamma = 0$. Since $I^2 + H$ is invertible implies that $I^2 + H''$ is invertible, it follows that $C'' = (I^2 + H'')^{-1}(\Gamma'')$. ■

The distribution constructed in Proposition 4.6 is the natural generalization of the horizontal distribution of the Levi-Civita connection to nonlinear geometries. It is easy to see that it agrees with the horizontal distribution of the Cartan connection in Finsler geometry, and in fact, when this distribution is used in Proposition 3.1, one obtains Rund and Cartan connections for an arbitrary semi-projectable triple.

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